Math 120A
Differential Geometry

Sample Midterm 2

Instructions: You have 50 minutes to complete the exam. There are five problems, worth a total of fifty points. You may not use any books or notes. Partial credit will be given for progress toward correct proofs.

Write your solutions in the space below the questions. If you need more space use the back of the page. Do not forget to write your name in the space below.

Name: ______________________________

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

Problem 1.

(a) [5pts.] Define an orientable surface.

Solution: A surface S is said to be orientable if there is an atlas $\mathcal{A}$ such that if Φ is a transition map between any two surface patches for S in $\mathcal{A}$ and $J(\Phi)$ is the Jacobian of Φ, then $\det(J(\Phi)) > 0$.

(b) [5pts.] Using any atlas you like, show directly that the unit sphere is orientable. (You are welcome to cite computations that have been assigned previously for the atlas you choose.)

Solution: We will modify the atlas for the unit sphere constructed on the homework using stereographic projection. We have two patches $\sigma_1 : \mathbb{R}^2 \to S - \{(0,0,1)\}$ given by the inverse of projection from the north pole and $\sigma_2 : \mathbb{R}^2 \to S - \{(0,0,-1)\}$ given by the inverse of projection from the south pole. On the homework we computed $\sigma_1^{-1} \circ \sigma_2^{-1} = \sigma_2 \circ \sigma_1^{-1} = \frac{1}{x^2+y^2}(x,y)$ as a map from $\mathbb{R}^2 - \{(0,0)\}$ to itself. The determinant of the Jacobian of this map is $\frac{-1}{(x^2+y^2)^2}$, and in particular is negative everywhere. Switch the two coordinates in σ_2 to obtain a new σ_2' which is equal to σ_2 composed with $(x,y) \to (y,x)$. This gives an atlas consisting of patches σ_1 and σ_2' where the Jacobian of the transition maps is everywhere positive.

[Also easy to do using the transition maps you computed for the six-patch atlas from the homework.]

Problem 2.

(a) [5pts.] Define an allowable surface patch.

Solution: We say $\sigma : U \to S \cap W$ is an allowable surface patch for S if$U$ is an open set in $\mathbb{R}^2$, W is an open set in $\mathbb{R}^3$, and σ is a homeomorphism which is smooth and has linearly independent partial derivatives at all points in its domain.

(b) [5pts.] Let $\gamma(s)$ be a regular unit-speed curve with nowhere-vanishing curvature. The tube of radius $a > 0$ around γ is parametrized by

$$\sigma(s,\theta) = \gamma(s) + a(\mathbf{n}(s)\cos\theta + \mathbf{b}(s)\sin\theta)$$

Show that σ is regular if the curvature κ of γ is less that $\frac{1}{a}$ everywhere.

Solution: We want to check that the partial derivatives of σ are linearly independent. We have

$$\begin{aligned}\sigma_s &= \mathbf{t} + a(-\kappa\mathbf{t} + \tau\mathbf{b}\cos\theta) + a(-\tau\mathbf{n}\sin\theta)\\ &= (1 - a\kappa\cos\theta)\mathbf{t} - a\tau\sin\theta\mathbf{n} + a\tau\cos\theta\mathbf{b}\\ &= (1 - a\kappa\cos\theta)\mathbf{t} - a\tau[\sin\theta\mathbf{n} - \cos\theta\mathbf{b}]\\ \sigma_\theta &= -a\sin\theta\mathbf{n} + a\cos\theta\mathbf{b}\\ &= -a[\sin\theta\mathbf{n} - \cos\theta\mathbf{b}]\end{aligned}$$

Since the cross produce of a vector with a linear multiple of itself is zero, the cross product of these vectors is

$$\begin{aligned}(1 - a\kappa\cos\theta)\mathbf{t} \times -a[\sin\theta\mathbf{n} - \cos\theta\mathbf{b}] &= -a(1 - a\kappa\cos\theta)[\sin\theta\mathbf{b} - \cos\theta(-\mathbf{n})]\\ &= -a(1 - a\kappa\cos\theta)[\sin\theta\mathbf{b} + \cos\theta\mathbf{n}]\end{aligned}$$

This is only zero if $1 - a\kappa\cos\theta = 0$, which can't happen if $\kappa < \frac{1}{a}$.

Problem 3.

(a) [5pts.] Define a generalized cylinder.

Solution: Let $\gamma(t)$ be a regular planar curve in $\mathbb{R}^3$ (typically in the xy-plane. A generalized cylinder is the union of all lines that perpendicular to the plane that pass through some $\gamma(t)$. [Also fine: the set of points $\{(\gamma_1(t), \gamma_2(t), z)\}$ for a regular curve γ.

(b) [5pts.] Prove that the solution set S of $3yz + 4xy + 9 = 0$ is a generalized cylinder. More specifically, what type is it?

Solution: We use diagonalization to put this quadric in standard form. Firstly, even cross terms are easier, so we multiply by two: $10yz + 6xy + 8 = 0$. Then our quadric is the solution set to $\mathbf{v}^t A\mathbf{v} + \mathbf{v}$ where

$$A = \begin{pmatrix} 0 & 4 & 0\\ 4 & 0 & 3\\ 0 & 3 & 0\end{pmatrix}$$

and $\mathbf{b} = (0, 4, 0)^t$. The characteristic polynomial of A is $-\lambda^3 + 25\lambda = -\lambda(\lambda^2 - 5)$, so the eigenvalues of A are 0 and $\pm\sqrt{5}$. Therefore we can apply a direct isometry to S to send it to the solution set of $\sqrt{5}x^2 - \sqrt{5}y^2 + 9 = 0$, which is a hyperbolic cylinder. (We don't have to actually find the orthogonal matrix we diagonalize by because $\mathbf{b}$ is zero.)

Problem 4.

(a) [5pts.] State the isoperimetric inequality.

Solution: Let $\gamma(t)$ be a regular simple closed curve in the plane. Then $A(\gamma) \leq \frac{\ell(\gamma)^2}{4\pi}$ with equality if and only if γ is a circle.

(b) [5pts.] Let $\gamma(t)$ be a regular simple closed curve which is area-maximizing for its length $\ell(\gamma)$. Describe the set of vertices of γ.

Solution: The curve γ is area-maximizing if and only if it is a circle, and in particular its curvature function κ_s is constant. Therefore $\dot{\kappa}_s \equiv 0$, so every point of γ is a vertex.

Problem 5.

The *genus one hyperelliptic involution* f is the map from the torus S to itself that takes $(x, y, z) \mapsto (x, -y, -z)$. Recall that one parametrization of the torus is

$$\sigma(\theta, \phi) = ((a + b\cos\theta)\cos\phi, (a + b\cos\theta)\sin\phi, b\sin\theta)$$

(a) [1pts.] How many points on the torus are fixed by f?

Solution: Four: $(\pm a \pm b, 0, 0)$.

(b) [5pts.] Given an arbitrary point $\mathbf{p} = (x, y, z)$ on the torus, find a matrix for the linear transformation $D_{\mathbf{p}}f$ in terms of appropriate bases for $TS_{\mathbf{p}}$ and $TS_{f(\mathbf{p})}$.

Solution: Notice that up to a multiple of $2\pi n$, the map f takes $\sigma(\theta, \phi) \to \sigma(-\theta, -\phi)$. Choose $\sigma : U_1 \to S$ such that $\mathbf{p} \in \sigma(U_1)$ and $\sigma : U_2 \to S$ such that $f(\mathbf{p}) \in \sigma(U_2)$, where in both cases σ is the parametrization above and U_1 and U_2 such that if $\theta, \phi \in U_1$), then $(-\theta, -\phi) \in U_2$, and σ is injective on both open sets. Without loss of generality $f(U_1) \subset U_2$. Then the connecting map $U_1 \to U_2$ is $(\theta, \phi) \to (-\theta, -\phi)$. Then the matrix of the determinant map in terms of the bases $\{\sigma_\theta(\theta, \phi), \sigma_\phi(\theta, \phi)\}$ for $T_{\mathbf{p}}S$ and $\{\sigma_\theta(-\theta, -\phi), \sigma_\phi(-\theta, -\phi)\}$ for $T_{f(\mathbf{p})}S$ is $-\operatorname{Id}$.

(c) [4pts.] Decide whether f is a diffeomorphism, a local diffeomorphism, or neither.

Solution: It is a diffeomorphism; in fact $f^2 = \mathrm{Id}$, so f is its own inverse.